

2.5.2. Steady-state Heat Conduction with heat generation.

The general equation: $\nabla^2 T(\vec{r}) + \frac{1}{k} g(\vec{r}) = 0$

Solution approach: <1> Finding a particular solution $T_p(\vec{r})$ that satisfies the original equation,

$$\nabla^2 T_p(\vec{r}) + \frac{1}{k} g(\vec{r}) = 0.$$

<2> Assuming $T(\vec{r}) = T_p(\vec{r}) + T_h(\vec{r})$, i.e., temperature can be written in two components — the particular solution and the rest.

Therefore, $T_h(\vec{r})$ satisfies:

$$\nabla^2 T_h(\vec{r}) = 0 \quad \text{homogeneous equation.}$$

Example: Consider a 2D steady-state heat conduction problem defined as follows, with heat generation $g(\vec{r}) = g_0$:

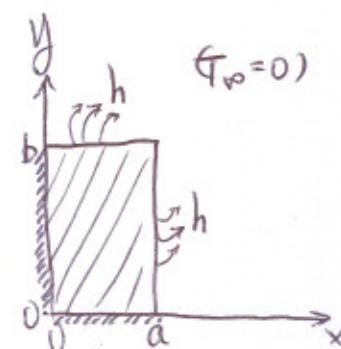
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{1}{k} g_0 = 0$$

$$\text{B.C. } \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0 \quad \leftarrow \text{homogeneous}$$

$$\left. \frac{\partial T}{\partial y} \right|_{y=0} = 0 \quad \leftarrow \text{homogeneous}$$

$$\left. \left(-k \frac{\partial T}{\partial x} \right) \right|_{x=a} = h T \Big|_{x=a} \quad \leftarrow \text{homogeneous}$$

$$\left. \left(-k \frac{\partial T}{\partial y} \right) \right|_{y=b} = h T \Big|_{y=b} \quad \leftarrow \text{homogeneous}$$



$$T = T(x, y)$$

Solution:

<1> finding a particular solution according to the form of heat generation function $g(\vec{r})$. There may be many choices of a particular solution, one of them is sufficient.

$$\underbrace{T(x, y)}_{\substack{\uparrow \\ \text{We pick one particular solution}}} = \underbrace{T_p(x)}_{\substack{\uparrow \\ \text{only the function of } x}} + T_h(x, y)$$

We pick one particular solution
only the function of x .

$T_p(x)$ must be the solution of the original heat conduction equation:

$$\frac{\partial^2 T_p(x)}{\partial x^2} + \frac{\partial^2 T_p(x)}{\partial y^2} + \frac{1}{k} g_0 = 0$$

i.e., $\frac{d^2 T_p(x)}{dx^2} + \frac{1}{k} g_0 = 0$

general solution: $T_p(x) = -\frac{g_0}{2k}x^2 + C_1x + C_2$ (C_1, C_2 arbitrary constant)

We pick $C_1 = C_2 = 0$ for simplicity

so:
$$T_p(x) = -\frac{g_0}{2k}x^2$$

Need only one particular solution

and:
$$T(x, y) = \underbrace{-\frac{g_0}{2k}x^2}_{\substack{\uparrow \\ \text{only one particular solution}}} + T_h(x, y)$$

<2> The original heat conduction equation becomes:

$$\underbrace{\frac{\partial^2 T_h(x,y)}{\partial x^2} + \frac{\partial^2 T_h(x,y)}{\partial y^2}}_0 = 0 \quad \leftarrow \text{homogeneous}$$

New boundary conditions are needed for $T_h(x,y)$.

By substituting $T = T_p + T_h$ into the original boundary conditions:

We have:

$$\left\{ \begin{array}{l} \left. \frac{\partial T_p(x)}{\partial x} \right|_{x=0} + \left. \frac{\partial T_h(x,y)}{\partial x} \right|_{y=0} = 0 \Rightarrow \left. \frac{\partial T_h}{\partial x} \right|_{x=0} = 0 \\ \left. \frac{\partial T_p(y)}{\partial y} \right|_{y=0} + \left. \frac{\partial T_h(x,y)}{\partial y} \right|_{x=0} = 0 \Rightarrow \left. \frac{\partial T_h}{\partial y} \right|_{y=0} = 0 \\ -k \left. \frac{\partial T_p(x)}{\partial x} \right|_{x=a} - k \left. \frac{\partial T_h(x,y)}{\partial x} \right|_{x=a} = h(T_p|_{x=a} + T_h|_{x=a}) \Rightarrow -k \left. \frac{\partial T_h}{\partial x} \right|_{x=a} = hT_h|_{x=a} - \left(\frac{h g_o}{2k} a^2 + g_o a \right) \\ -k \left. \frac{\partial T_p(y)}{\partial y} \right|_{y=b} - k \left. \frac{\partial T_h(x,y)}{\partial y} \right|_{y=b} = h(T_p|_{y=b} + T_h|_{y=b}) \Rightarrow -k \left. \frac{\partial T_h}{\partial y} \right|_{y=b} = hT_h|_{y=b} - \frac{h g_o}{2k} a^2 \end{array} \right.$$

There are two Nonhomogeneous boundary conditions for $T_h(x,y)$.

One of them can be eliminated by defining:

$$\overline{T}_h \equiv \frac{g_o}{2k} a^2 + \frac{g_o}{h} a$$

and: $\Theta_h(x,y) \equiv T_h(x,y) - \overline{T}_h$

Therefore,

$$\frac{\partial \theta_h(x,y)}{\partial x^2} + \frac{\partial^2 \theta_h(x,y)}{\partial y^2} = 0$$

B.C.

$$\left. \frac{\partial \theta_h}{\partial x} \right|_{x=0} = 0$$

$$\left. \frac{\partial \theta_h}{\partial y} \right|_{y=0} = 0$$

$$-k \left. \frac{\partial \theta_h}{\partial x} \right|_{x=a} = h \theta_h \Big|_{x=a}$$

$$-k \left. \frac{\partial \theta_h}{\partial y} \right|_{y=b} = h \theta_h \Big|_{y=b} + h \left(-\frac{q_0}{2k} x^2 + T_h^0 \right)$$

\leftarrow Nonhomogeneous

The original problem (Nonhomogeneous equation) is reduced to homogeneous equation with only one nonhomogeneous B.C.

3. Transient Conduction ($\frac{\partial}{\partial t} \neq 0$)

General Equation (3D),

$$\nabla^2 T(\vec{r}, t) + \frac{1}{k} g(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}$$

3.1. Homogeneous Transient Conduction Problem.

The homogeneous transient heat conduction problem —
homogeneous equation (no heat generation) with homogeneous
boundary conditions plus an initial condition —
Can be solved directly by the method of separation
of variables.

	$\nabla^2 T(\vec{r}, t) = \frac{1}{\alpha} \frac{\partial T(\vec{r}, t)}{\partial t}$	$(t > 0)$
B.C.	$k_i \frac{\partial T}{\partial n_i} \Big _{S_i} + h_i T \Big _{S_i} = 0$	$(t > 0)$
I.C.	$T(\vec{r}, t) \Big _{t=0} = F(\vec{r})$	$(t=0)$

To solve the homogeneous transient problem, we assume
the separation of temperature in the form of

$$T(\vec{r}, t) = \underbrace{\psi(\vec{r})}_{\substack{\uparrow \\ \text{function of } \vec{r}}} \Gamma(t)$$

By substituting $T = \Psi(\vec{r})\Gamma(t)$ into the original homogeneous heat conduction equation, we have:

$$\frac{\nabla^2 \Psi(\vec{r})}{\Psi(\vec{r})} = \frac{\frac{1}{\alpha} \frac{d\Gamma(t)}{dt}}{\Gamma(t)}$$

The left-hand side is a function of \vec{r} alone, and the right-hand side is a function of t alone, so the only way this equality holds is both sides are equal to the same constant.

$$\frac{\nabla^2 \Psi(\vec{r})}{\Psi(\vec{r})} = \frac{\frac{1}{\alpha} \frac{d\Gamma(t)}{dt}}{\Gamma(t)} = \mu \quad (\text{const.})$$

i.e.,
$$\begin{cases} \nabla^2 \Psi(\vec{r}) - \mu \Psi(\vec{r}) = 0 \\ \frac{d\Gamma(t)}{dt} = \alpha \mu \Gamma(t) \end{cases}$$

The ODE for $\Gamma(t)$ has the general solution:

$$\Gamma(t) = C e^{\alpha \mu t} \quad (C: \text{arbitrary constant})$$

In order to ensure that $\Gamma(t)$ does not diverge as time (t) increases indefinitely, μ must be negative,

i.e.: $\mu = -\lambda^2$

and: $\Gamma(t) = C e^{-\alpha \lambda^2 t} \quad (\alpha > 0)$

Note: if $\mu=0$, $\Gamma(t)$ will be a constant, so the problem is not transient.

The resulting equation for $\Psi(\vec{r})$ becomes

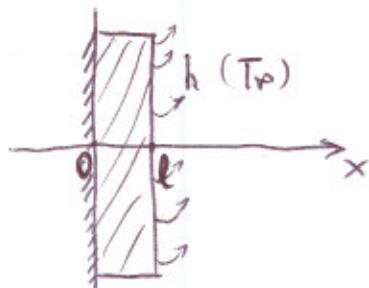
$$\left\{ \begin{array}{l} \nabla^2 \Psi(\vec{r}) + \lambda^2 \Psi(\vec{r}) = 0 \\ k_i \frac{\partial \Psi}{\partial n_i} \Big|_{S_i} + h_i \Psi \Big|_{S_i} = 0 \end{array} \right. \quad \begin{array}{l} \text{Helmholtz equation} \\ \text{on boundary } S_i \end{array}$$

3.2. Separation of Variables — Rectangular System

* Example 1.

Consider a flat plate of thickness l , initially at temperature T_0 . At $t=0$, the plate is suddenly immersed into a fluid of temperature T_∞ and heat is transferred to fluid at one surface, with the other surface insulated. Find the temperature distribution at any time.

The complete problem. (1D)



$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T(x,t)}{\partial t}$
B.C. $\left. \frac{\partial T}{\partial x} \right _{x=0} = 0$
$\left. -k \frac{\partial T}{\partial x} \right _{x=l} = h(T _{x=l} - T_\infty)$
I.C. $T _{t=0} = T_0$

Solution:

As one B.C. is nonhomogeneous ($T_{\infty} \neq 0$), we can define:

$$\theta \equiv T(x,t) - T_{\infty}$$

Therefore:

$\frac{\partial^2 \theta(x,t)}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta(x,t)}{\partial t}$ B.C. $\left. \frac{\partial \theta}{\partial x} \right _{x=0} = 0$ I.C. $\left. \theta \right _{t=0} = T_0 - T_{\infty}$	\leftarrow homogeneous \leftarrow homogeneous
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① Separation of $\theta(x,t)$:

Assume: $\theta(x,t) \equiv \underline{X}(x) \underline{\Gamma}(t)$

Therefore: $\begin{cases} \underline{X}''(x) + \lambda^2 \underline{X}(x) = 0 \\ \text{and: } \underline{\Gamma}(t) = C e^{-\alpha \lambda^2 t} \end{cases}$ \leftarrow this is the only possibility!

② Solving ODE for $\underline{X}(x)$:

$$\underline{X}''(x) + \lambda^2 \underline{X}(x) = 0$$

B.C. $\left. \frac{d\underline{X}}{dx} \right|_{x=0} = 0$

$$\left. -K \frac{d\underline{X}}{dx} \right|_{x=l} = h \underline{X} \Big|_{x=l}$$

The general solution for $\underline{X}(x)$ is:

$$\underline{X}(x) = A \cos \lambda x + B \sin \lambda x$$

Impose B.C. $\frac{dX}{dx} \Big|_{x=0} = 0$

$$\text{So: } \frac{dX}{dx} \Big|_{x=0} = -\lambda A \sin \lambda x \Big|_{x=0} + \lambda B \cos \lambda x \Big|_{x=0} = \lambda B = 0$$

Therefore: $B=0$

$$\text{And: } \underline{\underline{X(x) = A \cos \lambda x}}$$

Impose B.C. $\rightarrow k \frac{dX}{dx} \Big|_{x=l} = h \cdot X \Big|_{x=l}$

$$\text{So: } +k \lambda A \sin \lambda x \Big|_{x=l} = h A \cos \lambda x \Big|_{x=l}$$

$$\text{i.e.: } A(h \cos \lambda l - k \lambda \sin \lambda l) = 0$$

If $A=0 \Rightarrow X(x)=0 \Rightarrow$ not a meaningful solution.

$$\text{Therefore: } h \cos \lambda l - k \lambda \sin \lambda l = 0$$

$$\text{or: } \underline{\underline{\cot(\lambda l) = \frac{k}{h} \lambda}}$$

i.e., λ can only take certain values, as determined by this equation, $\lambda = \lambda_n$ with $n=1, 2, \dots$.

For each n :

$$\underline{\underline{X_n(x) = A \cos \lambda_n x}}$$

③ Making the final solution:

$$\underline{\underline{\theta(x,t) = \sum_{n=1}^{\infty} A_n \cos \lambda_n x e^{-\alpha \lambda_n^2 t}}}$$

④ Determining unknown coefficient.

Applying initial condition, $\theta|_{t=0} = T_0 - \bar{T}_0$

$$\text{So, } T_0 - \bar{T}_0 = \sum_{n=1}^{\infty} A_n \cos \lambda_n x$$

multiplying each side with $\cos \lambda_m x$ and integrating over $(0, l)$:

$$\int_0^l (T_0 - \bar{T}_0) \underline{\cos \lambda_m x} dx = \int_0^l \sum_{n=1}^{\infty} A_n \cos \lambda_n x \underline{\cos \lambda_m x} dx$$

Using the orthogonal property of $\cos \lambda_n x$:

We have:

$$A_n = \frac{(T_0 - \bar{T}_0) \int_0^l \cos \lambda_n x dx}{\int_0^l \cos^2 \lambda_n x dx}$$

$$= (T_0 - \bar{T}_0) \frac{\frac{\sin \lambda_n l}{\lambda_n}}{\frac{1}{2} \left(l + \frac{\sin \lambda_n l \cos \lambda_n l}{\lambda_n} \right)}$$

$$= (T_0 - \bar{T}_0) \frac{2 \sin \lambda_n l}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l}$$

$$\text{So, } \theta(x, t) = (T_0 - \bar{T}_0) \underbrace{\sum_{n=1}^{\infty} \frac{2 \sin \lambda_n l \cos \lambda_n x}{\lambda_n l + \sin \lambda_n l \cos \lambda_n l} e^{-\alpha_n^2 t}}$$

$$(\text{Note: } T(x, t) = \theta(x, t) + \bar{T}_0)$$